Тема 1 № 6

$u_{xx} + y \cdot u_{yy} = 0$

$B^2 - AC = -y$

В области $y>0$ эллиптический тип

В области $y<0$ гиперболический тип

Тема 1 № 9

$x \cdot u_{xx} + y \cdot u_{yy} = 0$

$B^2 - AC = -x \cdot y$

В областях $x>0, y>0$ и $x<0, y<0$ эллиптический тип

В областях $x>0, y<0$ и $x<0$ $y>0$ гиперболический тип

Тема 1 № 13

$y^2 \cdot u_{xx} + x^2 \cdot u_{yy} = 0$

$B^2 - AC = -x^2 \cdot y^2$

В областях $\{x>0, y>0\}$, $\{x<0, y>0\}$, $\{x>0, y<0\}$, $\{x<0, y<0\}$ эллиптический тип

Тема 1 № 25

$u_{xx} - 2x \cdot u_{xy} + \sin x = 0$

$(-2x)^2 - 1 \cdot 0 = 4x^2$

В областях $\{x>0\}$, $\{x<0\}$ гиперболический тип

Тема 11 № 1

1. $u_t = a^2 u_{xx}$

$a^2 u_{xx} + 0 \cdot u_{xt} + 0 \cdot u_{tt} - u_t = 0$

$B^2 - AC = 0$ - пар. тип

$dt = \frac{B^2 \pm \sqrt{B^2 - AC}}{A} dx = 0\, dx$

$t = C$ - характеристики (1 семейство)

2. $u_{xx} + u_{yy} = 0$

$B^2 - AC = -1 < 0$ - эл. тип

$dy = \frac{B^2 \pm \sqrt{B^2 - AC}}{A} dx$

$dy = \frac{0 \pm i}{1} dx = \pm i\, dx$

$y = \pm i x + C$

3. $u_{tt} = a^2 \cdot u_{xx}$ $\quad a^2 u_{xx} - u_{tt} = 0.$

$B^2 - AC = a^2 > 0$ - гип. тип.

$dt = \frac{\pm\sqrt{a^2}}{a^2} dx$

$dt = \pm \frac{1}{a} dx$

$t = \pm \frac{1}{a} x + C$

$\begin{cases} C = t - x/a \\ C = t + x/a \end{cases}$ - характеристики (2 семейства)

11. $x^2 \cdot u_{xx} - y^2 u_{yy} = 0$

$B^2 - AC = -x^2 \cdot (-y^2) = x^2 \cdot y^2 > 0$ – гиперболический тип

4 области

$x>0,\ y>0$

$dy = \dfrac{B \pm \sqrt{B^2 - AC}}{A} dx = \dfrac{\pm x \cdot y}{x^2} dx$

$dy = \pm \dfrac{y}{x} dx$

$x>0,\ y<0$

$dy = \mp \dfrac{xy}{x^2} dx$

$dy = \mp \dfrac{y}{x} dx$

$x<0,\ y>0$

$dy = \mp \dfrac{xy}{x^2} dx$

$dy = \mp \dfrac{y}{x} dx$

$x<0,\ y<0$

$dy = \pm \dfrac{x \cdot y}{x^2} dx$

$dy = \pm \dfrac{y}{x} dx$

$\dfrac{dy}{y} = \pm \dfrac{dx}{x}$

во всех областях хар-ки одинаковые

$\ln y = \pm \ln x + C$

$$\left[\begin{array}{l} y = \dfrac{C}{x} \iff xy = C \\ y = \cancel{x}C \iff \dfrac{y}{x} = C \end{array}\right.$$

$\xi = xy \qquad \eta = \dfrac{y}{x}$

$y = \sqrt{\xi\eta} \qquad x = \sqrt{\dfrac{\xi}{\eta}}$

$\xi_x = y \quad \xi_y = x \quad \xi_{xx} = 0 \quad \xi_{yy} = 0 \quad \xi_{xy} = 1$

$\eta_x = -\dfrac{y}{x^2} \quad \eta_y = \dfrac{1}{x} \quad \eta_{xx} = \dfrac{2y}{x^3} \quad \eta_{yy} = 0 \quad \eta_{xy} = -\dfrac{1}{x^2}$

$x^2 \,|\; u_{xx} = u_{\xi\xi} \cdot y^2 + 2u_{\xi\eta} \cdot y\left(-\dfrac{y}{x^2}\right) + u_{\eta\eta}\left(-\dfrac{y}{x^2}\right)^2 + u_\xi \cdot 0 + u_\eta\left(\dfrac{2y}{x^3}\right)$

$-y^2 \,|\; u_{yy} = u_{\xi\xi}\, x^2 + 2u_{\xi\eta}\, x \dfrac{1}{x} + u_{\eta\eta}\left(\dfrac{1}{x}\right)^2 + u_\xi \cdot 0 + u_\eta \cdot 0$

$0 = 0 + (-4y^2) u_{\xi\eta} + 0 + u_\eta\left(\dfrac{2y}{x^3}\right)$

$= 2 \cdot \dfrac{y}{x} \cdot \dfrac{1}{x^2}$

$-4(\xi\eta)\, u_{\xi\eta} + u_\eta\left(2 \cdot \eta\left(\dfrac{1}{\xi/\eta}\right)\right) = 0$

$-4\xi\eta\, u_{\xi\eta} + u_\eta \cdot 2 \cdot \dfrac{\eta^2}{\xi} = 0$

$$\boxed{u_{\xi\eta} = \frac{1}{2} \frac{\eta}{\xi^2} u_\eta}$$

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Тема 11 № 8

14. $y^2 \cdot u_{xx} + 2xy \cdot u_{xy} + x^2 \cdot u_{yy} = 0$

$B^2 - AC = (xy)^2 - y^2 \cdot x^2 = 0$ – параболический тип

$$dy = \frac{B \pm \sqrt{B^2 - AC}}{A} dx$$

$$dy = \frac{xy}{y^2} dx \quad \Rightarrow \quad dy = \frac{x}{y} dx$$

$y\,dy = x\,dx$

$y^2 = x^2 + C$

$\xi = x^2 - y^2$ $\qquad \xi_x = 2x \quad \xi_y = -2y \quad \xi_{xx} = 2 \quad \xi_{yy} = -2 \quad \xi_{xy} = 0$

Возьмем $\eta = y^2 \in C^1(\mathbb{R}^2)$ $\qquad \eta_x = 0 \quad \eta_y = 2y \quad \eta_{xx} = 0 \quad \eta_{yy} = 2 \quad \eta_{xy} = 0$

$$\begin{vmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 0 \\ -2y & 2y \end{vmatrix} = 4xy \neq 0 \text{ на } R \setminus \{\{x=0\} \cup \{y=0\}\})$$

$$\begin{array}{l|l} y^2 & u_{xx} = u_{\xi\xi}(2x)^2 + 2u_{\xi\eta} \cdot 4x \cdot 0 + u_{\eta\eta} \cdot 0 + u_\xi \cdot 2 + u_\eta \cdot 0 \\ x^2 & u_{yy} = u_{\xi\xi}(-2y)^2 + 2u_{\xi\eta} \cdot (-2y)2y + u_{\eta\eta}(2y)^2 + u_\xi \cdot (-2) + u_\eta \cdot 2 \\ 2xy & u_{xy} = u_{\xi\xi}\, 2x(-2y) + u_{\xi\eta}(2x \cdot 2y + (-2y) \cdot 0) + u_{\eta\eta} \cdot 0 \cdot 2y + u_\xi \cdot 0 + u_\eta \cdot 0 \end{array}$$

$$0 = u_{\xi\xi}(4x^2y^2 + 4x^2y^2 - 8x^2y^2) + u_{\xi\eta}(-8x^2y^2 + 8x^2y^2) + u_{\eta\eta}\, 4y^2 + u_\xi(2-2) + u_\eta \cdot 2$$

$$0 = u_{\eta\eta}\, 4y^2 + u_\eta \cdot 2$$

$$0 = u_{\eta\eta} \cdot 4\eta + u_\eta \cdot 2$$

$$u_{\eta\eta} + \frac{u_\eta}{2\eta} = 0$$

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№1

$$\begin{cases} u_{tt} = b^2 u_{xx} & 0 < x < \pi \quad t > 0 \\ u(0,t) = 0 \\ u_x(\pi, t) = 0 \\ u(x,0) = \sin\frac{3x}{2} + \sin\frac{5x}{2} \\ u_t(x,0) = b\sin\frac{7x}{2} \end{cases}$$

$u(x,t) = X(x)\cdot T(t)$

$\lambda_n = \left(\frac{1}{2} + n\right)^2$

$X_n = \sin(\sqrt{\lambda_n}\, x)$

$T_n = A_n \sin(b\sqrt{\lambda_n}\, t) + B_n \cos(b\sqrt{\lambda_n}\, t)$

$u(x,t) = \sum\limits_{n=0}^{\infty} \sin(\sqrt{\lambda_n}\, x)\left(A_n \sin(b\sqrt{\lambda_n}\, t) + B_n \cos(b\sqrt{\lambda_n}\, t)\right)$

$u(x,0) = \sum\limits_{n=0}^{\infty} B_n \sin\left(\left(\frac{1}{2}+n\right)\cdot x\right) = \sin\frac{3x}{2} + \sin\frac{5x}{2}$

$B_1 = 1 \quad B_2 = 1 \quad b_i = 0 \quad i \neq \overline{1,2} \quad i \in \mathbb{N} \cup \{0\}$

$u_x(x,0) = \sum\limits_{n=0}^{\infty} A_n b\sqrt{\lambda_n} \sin\left(\left(\frac{1}{2}+n\right)x\right) = b\sin\frac{7x}{2}$

$A_3 = \frac{1}{\frac{1}{2}+3} = \frac{2}{7} \qquad A_i = 0 \quad i \neq 3 \quad i \in \mathbb{N} \cup \{0\}$

$u(x,t) = \sin\frac{3x}{2}\cos\left(\frac{b3t}{2}\right) + \sin\frac{5x}{2}\cos\frac{5bt}{2} + \frac{2}{7}\cdot\sin\frac{7x}{2}\cdot\sin\frac{7bt}{2}$

№2

$$\begin{cases} u_{tt} = a^2 u_{xx} & 0 < x < \pi \\ u_x(0,t) = 0 \\ u(\pi,t) = 0 \\ u(x,0) = \cos\frac{3x}{2} + \cos\frac{7x}{2} \\ u_t(x,0) = a\cos\frac{5x}{2} \end{cases}$$

$u(x,t) = X(x)\cdot T(t)$

$X_n = \cos(\sqrt{\lambda_n}\, x) \qquad \lambda_n = \left(\frac{1}{2}+n\right)^2$

$T_n = A_n \sin(a\sqrt{\lambda_n}\, t) + B_n \cos(a\sqrt{\lambda_n}\, t)$

$u(x,t) = \sum\limits_{n=0}^{\infty} \cos(\sqrt{\lambda_n}\, x)\left(A_n \sin(a\sqrt{\lambda_n}\, t) + B_n \cos(a\sqrt{\lambda_n}\, t)\right)$

$u(x,0) = \sum\limits_{n=0}^{\infty} \cos(\sqrt{\lambda_n}\, x)\cdot B_n = \sum\limits_{n=0}^{\infty} B_n\cdot\cos\left(\left(\frac{1}{2}+n\right)x\right) = \cos\frac{3x}{2} + \cos\frac{7x}{2}$

$B_1 = 1 \quad B_3 = 1 \quad B_i = 0 \quad i \in \mathbb{N} \cup \{0\} / \{1,3\}$

$u_t(x,0) = \sum\limits_{n=0}^{\infty} \cos(\sqrt{\lambda_n}\, x) A_n a\sqrt{\lambda_n} = \sum\limits_{n=0}^{\infty} A_n a\left(\frac{1}{2}+n\right)\cdot\cos\left(\left(\frac{1}{2}+n\right)x\right) = a\cos\frac{5x}{2}$

$A_2 = \frac{1}{\frac{1}{2}+2} = \frac{2}{5} \qquad A_i = 0 \quad i \in \mathbb{N} \cup \{0\} / \{2\}$

$u(x,t) = \cos\frac{3x}{2}\cdot\cos\left(\frac{3at}{2}\right) + \cos\left(\frac{7x}{2}\right)\cdot\cos\left(\frac{7at}{2}\right) + \frac{2}{5}\cdot\cos\frac{5x}{2}\cdot\sin\left(\frac{5at}{2}\right)$

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№ 4

$$\begin{cases} u_{tt} = u_{xx} + \cos t & 0 \le x < \pi \quad t > 0 \\ u(0,t) = 0 \\ u(\pi, t) = 0 \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$$

Ш-Л $\lambda_n = n^2 \qquad X_n = \sin(nx) \qquad \|X_n\|^2 = \int_0^\pi \sin^2(nx)\,dx = \int_0^\pi \frac{1-\cos 2nx}{2}\,dx = \frac{\pi}{2}$

$$f_n = \frac{2}{\pi}\int_0^\pi \cos t \cdot \sin(nx)\,dx =$$

$$= \frac{2}{\pi}\cos t \left(-\frac{\cos(nx)}{n}\right)\Big|_0^\pi = \frac{2}{\pi n}\cos t\,(1-(-1)^n) = \begin{cases} 0 & n = 2k \\ \frac{4}{\pi n}\cos t & n = 2k+1 \end{cases}$$

T_n: $\begin{cases} T_n'' = -n T_n + f_n \\ T_n(0) = 0 \\ T_n'(0) = 0 \end{cases}$

В случае $f_n = 0 \qquad T_n = 0 \quad \Rightarrow T_{2k} = 0$

$n = 2k+1$ $$T_{2k+1} = \frac{1}{2k+1}\int_0^t \sin((2k+1)(t-\tau))\frac{4}{\pi(2k+1)}\cos\tau\,d\tau =$$

$$= \frac{4}{\pi(2k+1)^2}\int_0^t \frac{1}{2}\left(\sin((2k+1)(t-\tau)-\tau) + \sin((2k+1)(t-\tau)+\tau)\right)d\tau =$$

$$= \frac{4}{\pi((2k+1)^2}\,\frac{1}{2}\left[\frac{-\cos((2k+1)(t-\tau)-\tau)}{-2k-1-1} + \frac{-\cos((2k+1)(t-\tau)+\tau)}{-2k-1+1}\Big|_0^t\right] =$$

$$= \frac{4}{\pi(2k+1)^2}\cdot\frac{1}{2}\left[\frac{\cos(-t)}{+2k+2} + \frac{\cos t}{2k} - \frac{\cos((2k+1)t)}{2k+2} - \frac{\cos((2k+1)t)}{2k}\right] =$$

$$= \frac{4}{\pi(2k+1)^2}\cdot\frac{1}{2}\left[\frac{2\sin\left(\frac{(2k+1)t+t}{2}\right)\cdot\sin\left(\frac{(2k+1)t-t}{2}\right)}{2k+2} + \frac{2\sin\left(\frac{(2k+1)t+t}{2}\right)\cdot\sin\left(\frac{(2k+1)t-t}{2}\right)}{2k}\right.$$

$$= \frac{4}{\pi(2k+1)^2}\sin((k+1)t)\cdot\sin(kt)\left[\frac{2k+2k+2}{2k(2k+2)}\right] = \frac{2\sin((k+1)t)\cdot\sin(kt)}{\pi(k+1)(k)(2k+1)}$$

$$\Rightarrow u(x,t) = \sum_{k=0}^{\infty}\left[\frac{\sin((2k+1)x)\cdot\frac{2}{\pi}\cdot\sin((k+1)t)\sin(kt)}{(k+1)(k)(2k+1)}\right]$$